LAMINAR BOUNDARY LAYERS IN EXPONENTIAL FLOW ALONG AN INFINITE FLAT PLATE WITH UNIFORM SUCTION

K. S. PANDEY

MATHEMATICS SECTION, INSTITUTE OF TECHNOLOGY, B. H. U., VARANASI-5 INDIA

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ABSTRACT. The two-dimensional incompressible fluid flow problem along an infinite flat plate has been discussed, when the suction velocity, normal to the plate, is uniform and is directed towards it. Expressions for the velocity and skin-friction have been obtained in a non-dimensional form for two different cases : (1) exponentially increasing small perturbation, and (2) exponentially decreasing small perturbation. Here we have found that the exponentially increasing and decreasing cases do not give any back flow near the wall.

INTRODUCTION

Lighthill (1954) has considered the time-dependent viscous flow problem dealing with the effect of unsteady fluctuations of the free-stream velocity on the flow in the boundary layer of an incompressible fluid past two-dimensional bodies. He has obtained the solutions for the cases of low-and high-frequency approximations. Stuart (1955) has found some interesting features for an oscillatory flow over an infinite plate with uniform suction. The expression for velocity, obtained by him, shows that skin-friction fluctuations have a phase lead over the main-stream velocity fluctuation, the trend being in accord with Lighthill's (1954) theory, while the amplitude of the skin-friction fluctuation rises with frequency. He has further obtained that there is back flow in the case of high-frequency approximation. In the present paper, an attempt has been made to study the variation of skin-friction amplitude and velocity field with the variation of frequency. The external flow velocity has been taken as $U_0'(1+\epsilon e^{n't'})$, where U_0' is the mean of the main-stream velocity (large y') and ϵ is a small quantity. v_0' is taken to be the non-zero negative constant suction velocity. On analysis it has been found that the exponentially increasing and decreasing small perturbation cases do not give any back flow near the wall.

EQUATIONS OF MOTION

We consider a two-dimensional incompressible fluid flow along an infinite plane porous wall. The flow is independent of the distance parallel to the wall and the suction velocity v', normal to the wall, is directed towards it and is constant. The x'-axis is taken along the wall, y'-axis normal to the wall. Dashes denote dimensional quantities. Navier-Stokes equations and the equation of continuity become

$$\left.\begin{aligned} &\frac{\partial u'}{\partial t'}+v'\frac{\partial u'}{\partial y'}=-\frac{1}{\rho'}\frac{\partial p'}{\partial x'}+\nu\frac{\partial^2 u'}{\partial y'^2},\\ &\frac{\partial v'}{\partial t'}=-\frac{1}{\rho'}\frac{\partial p'}{\partial y'},\\ &\frac{\partial v'}{\partial y'}=0. \end{aligned}\right\} \qquad \cdots (1)$$

This system of equations is subject to the conditions

$$u'=0 \text{ at } y'=0 \text{ and } u'\to U'(t') \text{ at } y\to\infty,$$

$U'(t')$ being the velocity outside the boundary layer. Although $\frac{\partial v'}{\partial y'}=0$ in (1) shows that v' is a function of time only, we now further restrict consideration to the case of v' equal to a negative constant $(-v_0')$, from which it follows that p' is independent of y'. Consequently, $-\frac{1}{\rho'}\frac{\partial p'}{\partial x'}$ is equal to $\frac{dU'}{dt'}$, and the first equation of (1) becomes

$$\frac{\partial u'}{\partial t'}-v_0'\frac{\partial u'}{\partial y'}=\frac{dU'}{dt'}+\nu\frac{\partial^2 u'}{\partial y'^2}, \qquad (2)$$

subject to the conditions

$$u'=0 \text{ at } y'=0 \text{ and } u'\to U'(t') \text{ at } y'\to\infty.$$

Now, we introduce non-dimensional quantities defined by

$$y=\frac{y'|v_0'|}{\nu},\; t=\frac{v_0'^2 t'}{4\nu},\; n=\frac{4\nu n'}{v_0'^2},\; u=\frac{u'}{U_0'},\; U=\frac{U'}{U_0'}, \qquad (3)$$

where U_0' is a reference velocity and n' is the frequency. Equation (2) takes the non-dimensional form

$$\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}-\frac{1}{4}\frac{\partial u}{\partial t} \qquad \frac{1}{4}\frac{dU}{dt}, \qquad \cdots (4)$$

subject to the conditions

$$u=0 \text{ at } y=0 \text{ and } u\to U \text{ as } y\to\infty. \qquad \cdots (5)$$

1. *Exponentially Increasing Small Perturbation Case* : In this section, we consider the case in which the external flow velocity U follows the exponentially increasing small perturbation law.

Let us suppose, therefore, that

$$U = 1+\epsilon e^{nt}, \tag{6}$$

$$u = f_1(y)+\epsilon e^{nt} f_2(y). \tag{7}$$

Substituting in (4) and comparing harmonic terms, we get

$$\frac{d^2 f_1}{dy^2}+\frac{df_1}{dy} = 0, \tag{8}$$

$$\frac{d^2 f_2}{dy^2}+\frac{df_2}{dy}-\frac{n}{4}f_2 = -\frac{n}{4}, \tag{9}$$

subject to the conditions

$$\left.\begin{aligned} & f_1 = f_2 = 0 \text{ at } y = 0, \\ \text{and} \quad & f_1, f_2 \to 1 \text{ as } y \to \infty. \end{aligned}\right\} \tag{10}$$

The solutions of (8) and (9), satisfying (10), are

$$f_1 = 1-e^{-y},$$

$$f_2 = 1-e^{-hy},$$

where $h = \frac{1}{2}+\frac{1}{2}\sqrt{1+n}$.

Hence the velocity field in the boundary layer is given by

$$u(y, t) = 1-e^{-y}+\epsilon e^{nt}(1-e^{-hy}). \tag{11}$$

The non-dimensional skin-friction τ_0 is given by

$$= \frac{\tau'_0}{\rho' U'_0 |v'_0|} = \left(\frac{\partial u}{\partial y}\right)_{y=0} = 1+\epsilon h e^{nt}. \tag{12}$$

Now, for small values of the frequency parameter n, we have

$$h = 1+\frac{n}{4}-\frac{n^2}{16},$$

and

$$f_2 = 1-e^{-\left(1+\frac{n}{4}-\frac{n^2}{16}\right)y},$$

so that the velocity field and skin-friction are respectively given by

$$u(y, t) = 1-e^{-y}+\epsilon e^{nt}\left[1-e^{-\left(1+\frac{n}{4}-\frac{n^2}{16}\right)y}\right] \tag{13}$$

$$\tau_0 = \left(\frac{\partial u}{\partial y}\right)_{y=0} = 1+\epsilon e^{nt}\left(1+\frac{n}{4}-\frac{n^2}{16}\right) \tag{14}$$

For large values of the frequency parameter n, we ave

$$h=\frac{1}{2}+\frac{\sqrt{n}}{2}+\frac{1}{4\sqrt{n}}$$

and
$$f_2=1-e^{-\left(\frac{1}{2}+\frac{\sqrt{n}}{2}+\frac{1}{4\sqrt{n}}\right)y}.$$

Hence the velocity field and skin-friction are given by

$$u(y,t)=1-e^{-y}+\epsilon e^{nt}\left[1-e^{-\left(\frac{1}{2}+\frac{\sqrt{n}}{2}+\frac{1}{4\sqrt{n}}\right)y}\right] \quad \text{.. (15)}$$

$$\tau_0=\left(\frac{\partial u}{\partial y}\right)_{y=0}=1+\epsilon e^{nt}\left(\frac{1}{2}+\frac{\sqrt{n}}{2}+\frac{1}{4\sqrt{n}}\right) \quad \text{.. (16)}$$

respectively.

Figure 1 shows a fair agreement between the exact amplitude, h, and the low

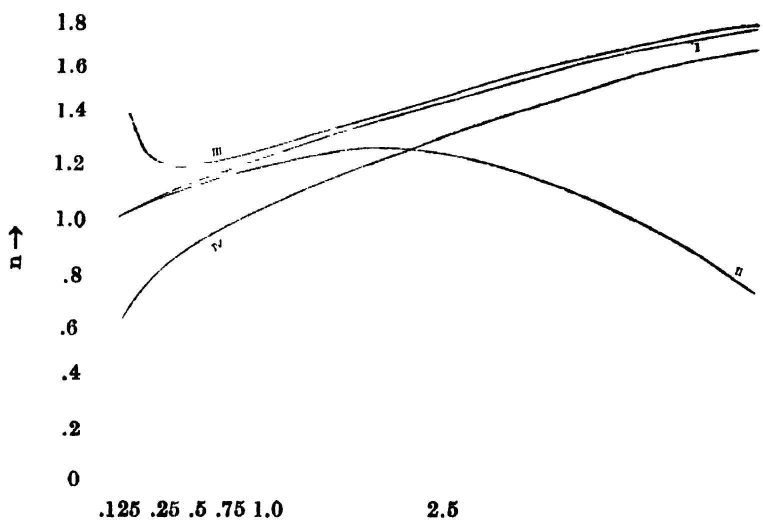

Figure 1. Skin-friction amplitude, h, againest the frequency parameter n. II, low-frequency approximation $h=1+\frac{n}{4}-\frac{n^2}{16}$; III, high-frequency approximation $h=\frac{1}{2}+\frac{\sqrt{n}}{2}+\frac{1}{4}\sqrt{n}$; I, exact value $h=\frac{1}{2}+\frac{1}{2}\sqrt{1+n}$; IV, high (rough)-frequency approximation $h=\frac{\sqrt{n}}{2}$

frequency approximation upto about $n=1$, though above $n=1$, the high-frequency approximation is in rather more satisfactory agreement with the exact value. The high (rough)-frequency approximation is not in agreement with the exact value.

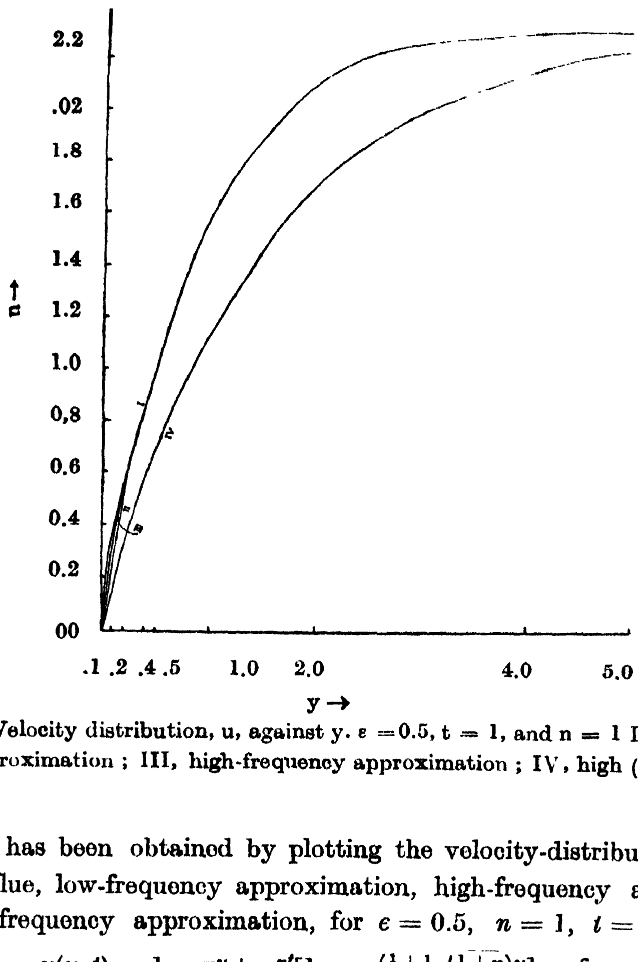

Figure 2. Velocity distribution, u, against y. $\varepsilon = 0.5$, $t = 1$, and $n = 1$ I, Exact ; II, low frequency approximation ; III, high-frequency approximation ; IV, high (rough) frequency approximation.

Figure 2 has been obtained by plotting the velocity-distribution u against y for exact value, low-frequency approximation, high-frequency approximation, high (rough)-frequency approximation, for $\epsilon = 0.5$, $n = 1$, $t = 1$, where, for exact value, $u(y, t) = 1-e^{-y}+\epsilon e^{nt}[1-e^{-(\frac{1}{2}+\frac{1}{2}\sqrt{1+n})y}]$, for low-frequency approximation,

$$u(y, t) = 1-e^{-y}+\epsilon e^{nt}\left[1-e^{-\left(1+\frac{n}{4}-\frac{n^2}{16}\right)y}\right]$$

for high-frequency approximation,

$$u(y, t) = 1-e^{-y}+\epsilon e^{nt}\left[1-e^{-\left(\frac{1}{2}+\frac{\sqrt{n}}{2}+\frac{1}{4\sqrt{n}}\right)y}\right]$$

and for high (rough)-frequency approximation,

$$u(y, t) = 1-e^{-y}+\epsilon e^{nt}\left[1-e^{-\frac{\sqrt{n}}{2}y}\right]$$

Figure 2 shows that the graphs of low and high-frequency approximations are in fair agreement with the graph of exact value near the wall. This graph also indicates that there is no back flow near the wall in the case of high-frequency approximation, whereas Stuart (1955) has shown that there is back flow near the wall in the case of high-frequency approximation for oscillatory flow.

2. *Exponentially Decreasing Small Perturbation Case*: In this section, we consider the case in which the external flow velocity follows the exponentially decreasing small perturbation law.

Let us suppose that

$$U = 1+\epsilon e^{-nt}, \qquad .. \quad (17)$$

$$u = f_1(y)+\epsilon e^{-nt} f_2(y). \qquad .. \quad (18)$$

Substituting in (4) and comparing harmonic terms, we get

$$\frac{d^2 f_1}{dy^2} + \frac{df_1}{dy} = 0, \qquad (19)$$

$$\frac{d^2 f_2}{dy^2} + \frac{df_2}{dy} + \frac{n}{4} f_2 = \qquad (20)$$

subject to the conditions

$$\left.\begin{aligned} f_1 = f_2 &= 0 \text{ at } y = 0, \\ \text{and} \qquad f_1, f_2 &\to 1 \text{ as } y \to \infty. \end{aligned}\right\} \qquad (21)$$

Solutions of (19) and (20), satisfying (21) are

$$f_1 = 1-e^{-y},$$

$$f_2 = 1-e^{-hy}$$

where $h = \frac{1}{2} + \frac{1}{2}\sqrt{1-n}$.

Hence the velocity field and skin-fraction are given by

$$u(y, t) = 1-e^{-y}+\epsilon e^{-nt}(1-e^{-hy}), \qquad (22)$$

$$\tau_0 = \left(\frac{\partial u}{\partial y}\right)_{y=0} \quad 1+\epsilon h e^{-nt}. \qquad (23)$$

Now, for small values of the frequency parameter n, we have

$$h = 1- \frac{n}{4} - \frac{n^2}{16},$$

and

$$f_2 = 1-e^{-\left(1-\frac{n}{4}-\frac{n^2}{16}\right)y},$$

so that the velocity field and skin-friction are respectively given by

$$u(y, t) = 1-e^{-y}+\epsilon e^{-nt}\left[1-e^{-\left(1-\frac{n}{4}-\frac{n^2}{16}\right)y}\right], \qquad .. \quad (24)$$

and $$\tau_0 = \left(\frac{\partial u}{\partial y}\right)_{y=0} = 1+\epsilon e^{-nt}\left(1-\frac{n}{4}-\frac{n^2}{16}\right). \qquad .. \quad (25)$$

For large values of the frequency parameter n, we have

$$h = \tfrac{1}{2} + i\left(\frac{\sqrt{n}}{2} - \frac{1}{4\sqrt{n}}\right),$$

$$f_2 = 1-e^{-\left\{\frac{1}{2}+i\left(\frac{\sqrt{n}}{2}-\frac{1}{4\sqrt{n}}\right)\right\}y},$$

where $$i = \sqrt{-1}\,.$$

Hence the velocity field and the skin-friction are given by

$$u(y,t) = 1-e^{-y}+\epsilon e^{-nt}\Big[1-e^{-y/2}\cos\left(\frac{\sqrt{n}}{2}-\frac{1}{4\sqrt{n}}\right)y$$

$$+ie^{-y/2}\sin\left(\frac{\sqrt{n}}{2}-\frac{1}{4\sqrt{n}}\right)y\Big], \qquad .. \quad (26)$$

$$\tau_0 = 1+\epsilon e^{-nt}\left[\tfrac{1}{2}+i\left(\frac{\sqrt{n}}{2}-\frac{1}{4\sqrt{n}}\right)\right] \qquad .. \quad (27)$$

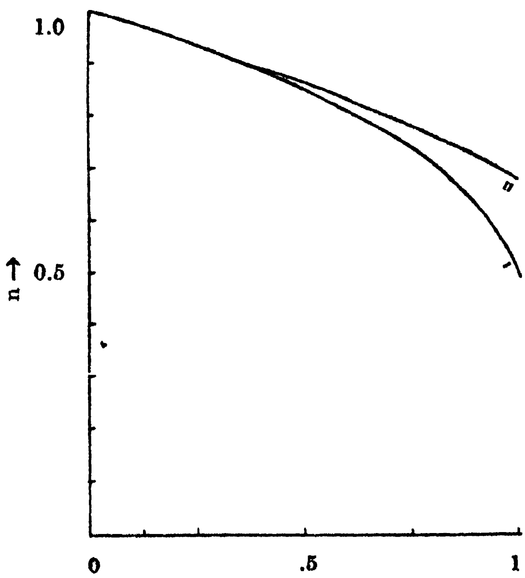

Figure 3. Skin-friction amplitude, h, against the frequency parameter n. II, low freequency approximation $h = 1 - \frac{n}{4} - \frac{n^2}{16}$; I exact value $h = \frac{1}{2} + \frac{1}{2}\sqrt{1-n}$

It is clear from figure 3 that the exact amplitude, h, is in fair agreement with the low-frequency approximation upto about $n = \frac{1}{2}$, but above $n = \frac{1}{2}$, the agreement begins to decrease.

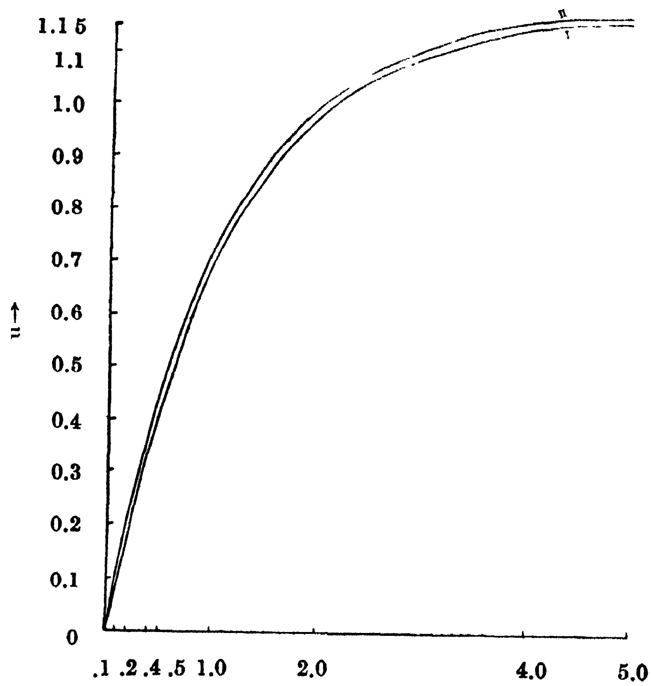

Figure 4. Velocity distribution u, against y. $\epsilon = 0.5$, $t = 1$, $n = 1$. I, Exact value, II, low-frequency approximation.

Figure 4 has been obtained by plotting the velocity distribution u against y for exact value and low-frequency approximation, for $\epsilon = 0.5$, $n = 1$ and $t = 1$, where, for exact value, $u(y, t) = 1 - e^{-y} + \epsilon e^{-nt}\left[1 - e^{-\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-n}\right)y}\right]$, and for low frequency approximation,

$$u(y, t) = 1 - e^{-y} + \epsilon e^{-nt}\left[1 - e^{-\left(1 - \frac{n}{4} - \frac{n^2}{16}\right)y}\right].$$

It is clear from figure 4 that there exists a fair agreement between the exact value and the low-frequency approximation near the wall, and also, there is no back flow near the wall.

Figure 5 shows the variation of the skin-friction amplitude $|h|$ with the variation of the frequency parameter n for high-frequency approximation. From the figure, it is clear that initially $|h|$ decreases with the increase of n upto about $n = \frac{1}{2}$, but above $n = \frac{1}{2}$, $|h|$ increases with the increase of n.

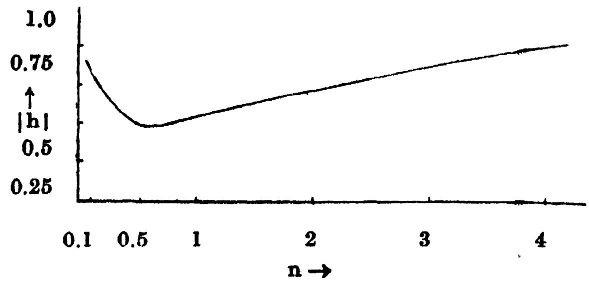

Figure 5. Skin-friction amplitude $|h|$ against frequency parameter n.

$$|h| = \left(\frac{n}{4} + \frac{1}{16n}\right)^{\frac{1}{2}}.$$

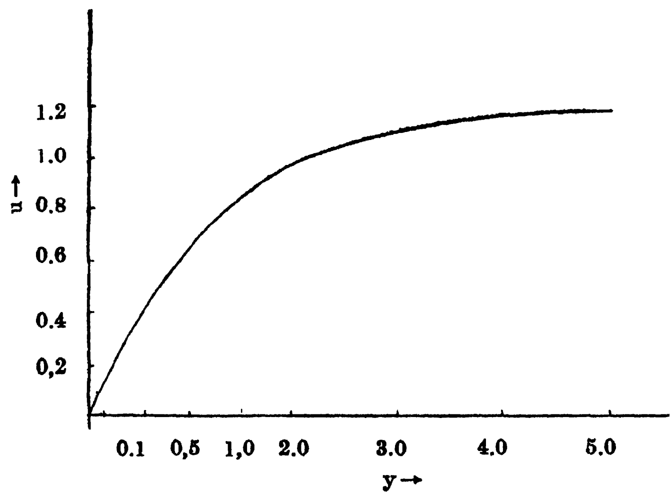

Figure 6. Transient velocity profile u, against y. $\epsilon = 0.5$, $n = 1$, $t = 1$.

$$u = 1 - e^{-y} + \epsilon e^{-nt}\left[1 - e^{-y/2}\cos\left(\frac{\sqrt{n}}{2} - \frac{1}{\sqrt{n}}\right)\right].$$

In figure 6, the transient velocity profile $u = 1 - e^{-y} + \epsilon e^{-nt}\left[1 - e^{-y/2}\cos\left(\sqrt{n} - \frac{1}{4\sqrt{n}}\right)y\right]$ is shown against the distance from the wall for the high-frequency approximation, where $\epsilon = 0.5$, $n = 1$, $t = 1$. From the figure it is clear that u increases as y increases and there is no back flow near the wall giving the similar result as that for the high-frequency approximation in the exponentially increasing small perturbation case.

CONCLUDING REMARKS

Comparing the results of exponentially increasing and exponentially decreasing small pertubation cases, we find that skin-friction amplitude in exponentially increasing case increases with the frequency, whereas, in the exponentially decreasing case, it decreases for the exact and low-frequency approximation but for high-frequency approximation, it increases with the frequency.

In both the cases (exponentially increasing and exponentially decreasing), we have found that there is no back flow near the wall.

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